A Modified Coefficient Ideal for Use with the Strict Transform

Anne Frühbis–Krüger
Institut f. Alg. Geometrie, Leibniz Universtät Hannover, Germany

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Abstract

Two main algorithmic approaches are known for making Hironaka's proof of resolution of singularities in characteristic zero constructive. Their main differences are the use of different notions of transforms during the resolution process and the different use of exceptional divisors in the descent in ambient dimension. In this article, we focus on the first difference. Only the approach using the weak transform has up to now been successfully used in implementations, because the other one requires an explicit stratification by the Hilbert-Samuel function at each step of the algorithm which is highly impractical due to the high complexity of the computation of such a stratification. In this article, a (hybrid-type) algorithmic approach is proposed which allows the use of the strict transform without the full impact of the complexity of the stratification by the Hilbert-Samuel function at each step of the desingularization process. This new approach is not intended to always be superior to the previously implemented one, instead it has its strengths precisely at the weak point of the other one and is thus a candidate to be joined with it by an appropriate heuristic.

1 Introduction

Existence and construction of a desingularisation has been one of the central questions in algebraic geometry since the end of the 19th century. In characteristic zero, it was proved in Hironaka's groundbraking work [7] in 1964, in which he also introduced standard bases w.r.t. a local ordering among other tools, whereas the case of positive characteristic is still open. Nevertheless, the interest in resolution of singularities in characteristic zero did not end at that time. Instead, the main interest only shifted toward the quest for a better, more constructive understanding of Hironaka's non-constructive proof in which the key is the choice of appropriate centers for the blowing ups which provide the desingularization. This development led to two main algorithmic approaches to the proof: one due to Bierstone and Milman (see e.g. [2]) using a stratification by the Hilbert-Samuel-function at the beginning of each choice-of-center

step and the strict transform as the corresponding notion of transform. The other one, due to Villamayor with many contributions and simplifications by others over the years (see e.g. [1], [4], [8]), does not need the Hilbert-Samuel function in the choice-of-center step, but pays for it by using the weak transform which (compared to the strict transform) picks up extra components lying inside the exceptional divisor at each blowing up.[1] This latter approach has been used in the two currently existing implementations, mainly because its invariant is more accessible to practical calculations (see e.g. [3] and [6]). These implementations, on the other hand, have brought certain questions and even conjectures, which had been treated purely theoretically up to that point, into the reach of computer experiments: among these e.g. explicit computation of the topological zeta-function in the quest for a counter example to the monodromy conjecture, treatment of singularities in singular learning machines and in hidden Markov models in algebraic statistics or systematic study of multiplier ideals for cases beyond plane curves (see e.g. [11], [10]). These applications also showed that there are classes of examples, where the current implementations are rather far from choosing the optimal one among different possible sequences of centers. Additionally, the orders/degrees of the higher order generators of the ideal in question tend to grow much faster for the weak transform than for the strict transform. As the order, Villamayor's main invariant, is blind to the order of higher order generators, this has positive influence on the algorithmic proof, but very negative one on the efficiency of the implemented algorithm.

Therefore it is a legitimate question to ask why the other approach has not been implemented up to now. Here the stratification by the Hilbert-Samuel function or more precisely the task of finding the stratum of maximal Hilbert-Samuel function turned out to be the crucial issue. Computer experiments (see e.g. [9]) showed that even in very careful implementations this step is far from being sufficiently efficient to be used in every choice-of-center step, because it involves a parametric standard basis calculation with as many parameters as there are variables in the basering. Additionally, the internal differences in the descent in dimension, which were already mentioned in a footnote above, turn out to be without significant impact on the overall efficiency, because the fewer blowing ups needed by Bierstone and Milman have to be paid for by the algorithmically more involved use of a special coordinate system and logarithmic derivatives. Thus a direct implementation of the Bierstone-Milman approach does not promise any improvement in speed or size as compared to the implemented algorithms. If, however, the calculation of the Hilbert-Samuel stratum can be avoided in a significant number of steps, this approach could again be an option. In this article, we follow this line of thought and provide an outline of an algorithm which does not compute the Hilbert-Samuel stra-

[1] This difference is not the only one between the two approaches. They also differ in their descent in dimension of the ambient space. For Villamayor's approach we can basically use any local coordinate system at a given point and use the usual notion of derivatives. Bierstone and Milman, on the other hand, pay special attention to a choice of the local coordinates, taking into account the exceptional divisors passing through the given point, and use logarithmic derivatives to ensure that they can factor out larger powers of the exceptional divisors after the descent in dimension.

tum in each step, but instead computes auxilliary ideals which provide the information, whether the Hilbert-Samuel function of the original ideal dropped, in terms of their order. The geometric idea behind this is to modify the original ideal by adding extra components which only emphasize the contribution by certain generators of the ideal without having negative impact on the descending induction on dimension of the ambient space, which is the key to Hironakas proof and all constructive variants of it.

In the section 3, we give the definition of our modified coefficient ideal and then apply it to the problem of resolution of singularities in the following section. It is important to observe at this point, that the modified coefficient ideal is just another way of stating that we pass through an intermediate auxilliary ideal before entering the first descent in dimension. Section 5 and 6 are then of a more practical nature, the first one discussing the computational issues of this new approach and the last one illustrating the construction on two simple explicit examples.

As the ideas leading to this article developed in several steps over time and underwent more than one metamorphosis before taking the shape in which they are presented here, I am indepted to many people for fruitful discussions, among them Bernard Teissier, Vincent Cossart, Herwig Hauser, Edward Bierstone, Roushdi Bahloul, Gerhard Pfister, Rocio Blanco and Patrick Popescu-Pampu. I'd also like to thank the Institut de Mathématiques de Jussieu for their hospitality, at which I was a visitor when my first ideas in this direction evolved.

2 Basic Definitions and Notations

In order to fix notation, we would like to first state the basic defintions and some selected properties of the invariant controlling the choice of centers. A section of just 1 or 2 pages can obviously not suffice to even give a brief introduction into the intricacies of resolution of singularities, instead we would like to point to more thorough discussions in section 4.2 of [5] from the practical point of view and in [4] embedded in a detailed treatement of the resolution process.

In the general setting for these definitions, W is a smooth equidimensional scheme over an algebraically closed field K of characteristic zero and $X \subset W$ a subscheme thereof. For the purpose of quickly fixing notation in this section, we immediately focus on one affine chart U with coordinate ring R and denote the maximal ideal at $x \in U$ by $\mathfrak{m}_x$; $x_1, \ldots, x_d$ are a local system of parameters of R at x.

The order of an ideal $I = \langle g_1, \ldots, g_r \rangle \subset R$ at a point $x \in U$ is defined as

$$ord_x(I) := \max\{m \in \mathbb{N} | \; I \subset \mathfrak{m}_x^m\};$$

the locus of order ≥ 2 of I can be computed as the vanishing locus of

$$\Delta(I) = \langle \{g_i | 1 \leq i \leq s\} \cup \{\frac{\partial g_i}{\partial x_j} | 1 \leq i \leq r, 1 \leq j \leq d\} \rangle,$$

the locus of order $\geq c$ as the one of $\Delta^{c-1}(I)$.
The Hilbert-Samuel function of R/I at x is defined as

$$\begin{array}{rcl} HS_x : \mathbb{N} & \longrightarrow & \mathbb{N} \\ s & \longmapsto & length(R/\mathfrak{m}_x^{s+1}), \end{array}$$

comparison is performed lexicographically. Locally at a point it can be computed by determining a standard basis at this point w.r.t. a local degree ordering and subsequent combinatorics on its leading ideal. Determining its locus of maximal value, however, is an expensive parametric Groebner basis calculation involving as many parameters as there are variables.
Having fixed notation for these invariants, we can now state the general structure of the invariant which controls the choice of center in the resolution process:

$$(ord \text{ or } HS, n; ord, n; ord, n; \dots)$$

where n denotes a count of certain exceptional divisors which is not going to have any impact on the considerations in this article. The semicolon in the invariant denotes the key step in the invariant, Hironaka's descent in dimension of the ambient space by means of a hypersurface of maximal contact. Hypersurfaces of maximal contact are defined by order 1 elements of $\Delta^{max_x ord_x(I)-1}(I)$, which satisfy certain normal crossing conditions; choosing a hypersurface of maximal contact can be interpreted as locally choosing a main variable. The order directly after the semicolon then denotes the order of an auxilliary ideal, the coefficient ideal[2], arising in the descent in dimension by suitably collecting the coefficients of the powers of the main variable. More precisely, let $Z = V(z)$ be a hypersurface of maximal contact for I at x, then the coefficient ideal of I w.r.t. Z can be computed as

$$Coeff_Z(I) = \sum_{k=0}^{ord_x(I)-1} I_k^{\frac{k!}{k-i}}$$

where I_k is the ideal generated by all polynomials which appear as coefficients of z^k in some element of I.
Having stated the basic notions necessary to fix notation for discussing algorithmic choice of center, we also need to briefly consider notions of transforms under blowing ups: Let $\pi : \tilde{U} \longrightarrow U$ be a blowing up at a center C with exceptional divisor E. Then the total transform of $X \subset U$ (defined by the ideal I_X) under the blowing up π is given by $I_X\mathcal{O}_{\tilde{U}}$. The strict and weak transform can then be computed as

$$I_{X,strict} = (I_X\mathcal{O}_{\tilde{U}} : I_E^\infty) \quad \text{and} \quad I_{X,weak} = (I_X\mathcal{O}_{\tilde{U}} : I_E^b)$$

where b is the largest integer such that $I_E^b \cdot (I_X\mathcal{O}_{\tilde{U}} : I_E^b) = I_X\mathcal{O}_{\tilde{U}}$.

[2]More precisely, the order of the non-monomial part of the coefficient ideal is taken after decomposing into a product of ideals consisting of a monomial part, i.e. a product of powers of the exceptional divisors, and a non-monomial part.

3 Defining a Modified Coefficient Ideal

In this section, we define our modified coefficient ideal at a point $w \in W$ in a very special situation first and subsequently study whether we can always pass from the general case to this particular setting. In the special situation, order reduction for the appropriately marked modified coefficient ideal is equivalent to a drop in the Hilbert-Samuel function of the original ideal under strict transform.

To this end, let $y_1, \ldots, y_n$ be a local system of parameters for $\mathcal{O}_{W,w}$ and let $\mathcal{I}_{X,w} = \langle f_1, \ldots, f_k \rangle$ be subject to the following conditions:

(1) $f_1, \ldots, f_k$ is a reduced standard basis of $\mathcal{I}_{X,w}$ with respect to a local degree ordering such that $y_1 > y_2 > \cdots > y_n$. We assume these to be numbered such that $d_1 \le d_2 \le \cdots \le d_k$ where $d_i = ord_w(f_i)$.

(2) There are integers $1 \le e_1 \le \cdots \le e_k \le n$ such that for each $1 \le i \le k$ all $V(y_l)$, $1 \le l \le e_i$, are hypersurfaces of maximal contact for the ideal

$$J_i = \langle \underline{y}^{\alpha^{(1)}} f_1, \ldots, \underline{y}^{\alpha^{(i-1)}} f_{i-1}, f_i, \ldots, f_k \mid \alpha^{(r)} \in I_{r,i} \forall 1 \le r \le i-1 \rangle$$

where the numbers e_i are maximal with this property and $I_{r,i}$ is the set of all multi-indices for which $\sum_{j=1}^{e_s} \alpha_j^{(r)} \ge max\{0, d_s - d_r\}$ for all $1 \le s < i$ and $|\alpha^{(r)}| = d_i - d_r$.

To be able to reference coefficients of each $f_s \in J_i$ w.r.t. monomials in the variables $y_1, \ldots, y_{e_i}$ separately, we write $f_s = \sum_\beta a_\beta^{(s,i)} \underline{y_{(i)}}^\beta$.

Definition 1 *The modified coefficient ideal of $\mathcal{I}_{X,w}$ is then defined as the usual coefficient ideal of the ideal J_k with respect to $Z = V(y_1, \ldots, y_{e_k})$. More explicitly,*

$$Coeff_Z^{new}(\mathcal{I}_{X,w}) = \sum_{j=0}^{d_k - 1} I_j^{\frac{d_k!}{d_k - j}}$$

where $I_j = \langle a_\beta^{(i,k)} \mid \sum_{l=1}^{e_k} \beta_l \le j - d_k + d_i \rangle$.

As will be discussed in detail in section 4, this modified coefficient ideal is in no way intended to be used in all descents in dimension of the ambient space in the computation of the value of the invariant. Instead, it only replaces the usual coefficient ideal in the descent of highest ambient dimension, accommodating for the use of the strict transform. In all further descents, we use again the usual coefficient ideal, which for the time being is the one in Villamayor's approach, although the approach could easily be changed to use the one of Bierstone and Milman.

Remark 2 *The use of a standard basis in the above definition is not necessary for the coefficient ideal itself: as all elements of a standard basis can be expressed as linear combinations of the original generators over the base ring, all contributions to the coefficient ideal already originate from these. The fact that we do not require identiy, but only $\le$ in the condition concerning the I_j again emphasizes this point of view. In the*

subsequent section, however, the standard basis property will be used when considering the effects of blowing up on the coefficient ideal, because this allows a significantly simpler treatment of the strict transform of $\mathcal{I}_{X,w}$.

The crucial issue about the above definition is its possible dependence on a particular choice of the local system of parameters. This, however, does not prevent its use in a resolution invariant as the following lemma observes:

Lemma 3 *Different choices of the $y_1, \ldots, y_n$ (subject to conditions (1) and (2)) affect neither the order of the modified coefficient ideal nor the orders of the subsequent (usual) coefficient ideals.*

This, obviously, implies that the order of the non-monomial part is also unaffected by those different choices, as the order of the monomial part is not sensitive to it.

Proof: First we observe that for J_k itself different choices of the hypersurfaces of maximal contact do not affect the orders of the subsequent coefficient ideals, since we are using precisely the usual Villamayor-style resolution invariant. What remains to be proved is that the values of the invariant coincide for any two ideals J_k and J'_k arising from the original ideal $\mathcal{I}_{X,w}$ as described above w.r.t. two different local systems of parameters[3] $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$. To this end, we first consider hypersurfaces of maximal contact for J_i which give rise to hypersurfaces of maximal contact for J_{i+1} and then proceed by comparing the (usual) coefficient ideal of J_k and J'_k with respect to suitably chosen flags.

<u>Step 1:</u> descent in ambient dimension for J_k

Let $x_1, \ldots, x_n$ be the regular system of parameters chosen in the construction of J_k and let $V(y)$ be a hypersurface of maximal contact for J_i for an arbitrary $1 \leq i < k$ (satisfying $y \in \Delta^{d_i-1}(J_i)$ and $y \not\equiv 0 \mod \mathfrak{m}^2_{W,w}$). Recalling that $\Delta^{d_i-1}(J_i)$ is generated by the generators of J_i and all their partial derivatives up to the (d_1-1)st ones, we now consider the analogously constructed set of generators of $\Delta^{d_{i+1}-1}(J_{i+1})$. For those elements originating from standard basis elements f_s with $s > e_i$, we find again all generators we already had plus additionally higher derivatives thereof; for the elements originating from an f_s, $s \leq e_i$, we now check the corresponding property by explicit calculation:

Let $g = \frac{\partial^{\gamma} f_s}{\partial y^{\gamma}}$ be an arbitrary element of $\Delta^{d_i-1}(J_i)$ and let $\tilde{\gamma} = (\gamma_1 + d_{i+1} - d_i, \gamma_2, \ldots, \gamma_n)$. Then

$$\frac{\partial^{\tilde{\gamma}} y_1^{d_{i+1}-d_i} f_s}{\partial y^{\tilde{\gamma}}} = \sum_{l=0}^{d_{i+1}-d_i} \binom{d_{i+1}-d_i}{l} \frac{\partial^l y_1^{d_{i+1}-d_i}}{\partial y_1^l} \cdot \frac{\partial^{(\tilde{\gamma}_1-l,\gamma_2,\ldots,\gamma_n)} f_s}{\partial y^{(\tilde{\gamma}_1-l,\gamma_2,\ldots,\gamma_n)}},$$

where the summand for $l = d_{i+1} - d_i$ is precisely the desired g up to a non-zero constant factor. But y_1 itself is an element of $\Delta^{d_i-1}(J_i)$ which implies that it can be written as a finite linear combination

$$y_1 = \sum_t a_t \frac{\partial^{\eta_t} f_{i_t}}{\partial y^{\eta_t}},$$

[3] These two systems of parameters are of course both subject to the conditions (1) and (2).

where the a_t are constants, $|\eta_t| \leq d_1 - 1$. As the appearing derivatives of the f_{i_t} for $i_t > e_i$ are in $\Delta^{d_{i+1}-1}(J_{i+1})$ by construction, we may assume without loss of generality that $i_t \leq e_i$ for all t. Replacing the f_{i_t} by $\frac{1}{(d_{i+1}-d_i)!} y_1^{d_{i+1}-d_i} \cdot f_{i_t}$ and the η_t by $\tilde{\eta_t}$, the corresponding linear combination yields:

$$\begin{aligned} \sum_t a_t \frac{1}{(d_{i+1}-d_i)!} \frac{\partial^{\tilde{\eta_t}} y^{d_{i+1}-d_i} f_{i_t}}{\partial y^{\tilde{\eta_t}}} &= \underbrace{\sum_t a_t \frac{\partial^{\eta_t} f_{i_t}}{\partial y^{\eta_t}}}_{=y_1} + y_1 \cdot c_2 \\ &= (1+c_2) \cdot y_1, \end{aligned}$$

where c_2 is a positive constant. Hence $y_1 \in \Delta^{d_{i+1}-1}(J_{i+1})$ which in turn implies by the above considerations that indeed $g \in \Delta^{d_{i+1}-1}(J_{i+1})$ as was to be proved.
This shows that indeed $\Delta^{d_i-1}(J_i)$ is contained in $\Delta^{d_{i+1}-1}(J_{i+1})$ which implies that $y_s \in \Delta^{d_{i+1}-1}(J_{i+1})$, $1 \leq s \leq e_i$ and hence proves the claim that $V(y_s)$ is a hypersurface of maximal contact for J_{i+1}.

Step 2: $y_i - x_i \in \mathfrak{m}_{W,w}^2$
Coming back to our original problem of comparing the coefficient ideals of J_k and J'_k, we split our considerations into two parts. In this first one, we assume that $g_l := y_l - x_l \in \mathfrak{m}_{W,w}^2$, and use that $x_1, \ldots, x_{e_k}$ give rise to hypersurfaces of maximal contact for J'_k and $x_1 - g_1, \ldots, x_{e_k} - g_{e_k}$ do so for J_k. A coordinate change replacing x_i by $x_i + g_i$ for all $1 \leq i \leq e_k$ transforms J_k into J'_k and $x_i - g_i$ into x_i, thus passing from one case to the other. Therefore the appearing orders of all subsequent coefficient ideals coincide.

Step 3: general case
By applying the construction of step 2 as a preparation step for one of the two systems of parameters and as a postprocessing step for the other in the general case, we may restrict our considerations to a linear change of coordinates such that both systems of parameters satisfy the conditions (1) and (2). In this case, we do not even need to worry about the orders, as the coefficients of the monomials of a given degree are only transformed into linear combinations thereof still in the same degree (according to appropriate matrices with constant entries having full rank). Hence the corresponding orders are unchanged.

q.e.d.

After studying the effects of different choices of the system of parameters on the resulting resolution invariant, we now discuss how to pass from the general situation to the special situation by a variant of the standard basis algorithm:

Let $\mathcal{I}_{X,w}$ be as above and let $d_1 := ord_w(\mathcal{I}_{X,w})$ be its order. Then we know that $\mathcal{I}_{X,w}$ contains at least one generator of order d_1 at w. We further know that $e_1 := dim_K(\Delta^{d_1-1}(\mathcal{I}_{X,w})/\mathfrak{m}_{W,w}^2) > 0$. Thus we can choose $y_1, \ldots, y_{e_1} \in \mathfrak{m}_{W,w}$ giving rise to a basis of this finite dimensional vector space and extend this to some local system of parameters. Expressing the

generators of $\mathcal{I}_{X,w}$ w.r.t. the new system of parameters, we enter the standard basis calculation choosing a local degree lexicographical ordering on the set of monomials in the chosen system of parameters. In the standard basis algorithm, we only treat s-polynomials of pairs with original leading monomials in degree d_1, postponing all calculations in higher degrees. After appropriate renumbering and interreduction of the generators, we may assume that $f_1, \ldots, f_i$ are precisely the standard basis elements of order $d_1 = \cdots = d_i$ and satisfy $LM(f_1) > LM(f_2) > \cdots > LM(f_i)$, where each of these leading monomials is some product of powers of $y_1, \ldots, y_{e_1}$.
We now set $e_j := e_1$ for all $1 \leq j \leq i$. From now on, we proceed degree by degree through the standard basis algorithm, adding new generators f_j to our evolving standard basis as necessary, defining the corresponding orders d_j accordingly and setting $e_j := e_1$, until a leading monomial appears which is not a product of the $y_1, \ldots, y_{e_1}$. We do not yet add this polynomial f_{s+1} to the evolving standard basis, because we first need to take care of adjusting our local system of parameters in the current degree d_l. To this end, we form a second ideal J by dropping the elements of our evolving standard basis from the set of generators of the ideal and subsequently adding all products $\underline{y}^\alpha f_j$, where f_j is one of the previously dropped elements of the evolving standard basis and $|\alpha| = d_l - d_j$. By construction, we now have $e_{s+1} := dim_K(\Delta^{d_1+|\alpha|-1}(J)/\mathfrak{m}^2_{W,w}) > e_s = e_1$. As before, we choose $y_{e_1+1}, \ldots, y_{e_r}$ such that $y_1, \ldots, y_{e_r}$ give rise to a basis of this vector space, and extend it to a local system of parameters. From here on, we proceed as before degree by degree through the standard basis calculation extending the standard basis, defining new d_l and e_l as needed and passing to a new local system of parameters according to the above construction when appropriate.

Remark 4 *In an affine setting, a reduced standard basis can be computed in family along each given connected component of a given stratum w.r.t. the Hilbert-Samuel function. This fact will be very useful for the computational point of view, which we will consider later on in section 5.*

4 The Modified Coefficient Ideal and the Resolution Process

From the construction introduced in the previous section, it is obvious that a center determined in this way will always be contained in the locus of maximal value of the Hilbert-Samuel function of $\mathcal{I}_X$. To study the use of the modified coefficient ideal in the resolution process, we would like to compare the effects of a blowing up in such a center to our original $\mathcal{I}_X$ and to the auxilliary ideals J_k.

Lemma 5 *The maximal value of the Hilbert-Samuel function of the strict transform of $I_{X,w}$ is smaller than the original one at w if and only if the maximal order of the weak transform of J_k is strictly less than $d_k = ord_w(J_k)$.*

Proof: If the maximal value of the Hilbert-Samuel function of the strict transform has decreased under the current blowing up, let us consider an arbitrary point w_1 in the preimage of w under the blowing up. There is at least one element, say g, of the reduced standard basis whose order has dropped, because the use of the reduced standard basis allows us to compute the strict transform by considering the strict transforms of the elements of the standard basis. But this implies that the order of the strict transform of each generator $y^\alpha g$ of J_k is strictly less than d_k and hence the order of the weak transform of J_k can no longer be d_k.
To prove the converse, let us now assume that the maximal order of the weak transform of J_k is no longer d_k and let us fix an arbitrary point w_1 in the preimage of w under the blowing up. There is at least one generator of J_k, say $y^\alpha h$, whose strict transform has order less than d_k at w_1. A priori two situations may have occured: $ord_{w_1}(h_{strict}) < ord_w(h)$, which directly implies a drop in the maximal value of the Hilbert-Samuel function under this blowing up, or $ord_{w_1}((y^\alpha)_{strict}) < |\alpha|$. In this second case, we can obviously find one y_j whose strict transform has order zero. But this y_j was chosen as a local equation of a hypersurface of maximal contact for $\mathcal{I}_{X,w}$ and hence occurs as a factor of at least one term of lowest order in at least one standard basis element, say g of $\mathcal{I}_{X,w}$. But this implies $ord_{w_1}(g_{strict}) < ord_w(g)$ which again corresponds to a decrease of the Hilbert-Samuel function as claimed.

q.e.d.

Lemma 6 *If the maximal value of the Hilbert-Samuel function is unchanged under blowing up, i.e. if $HS_w(\mathcal{I}_X) = HS_{w_1}((\mathcal{I}_X)_{strict})$, the weak transform of J_k coincides with the newly constructed $\tilde{J}_k$ of the strict transform of $\mathcal{I}_{X,w}$.*

Proof: As the construction of the J_k uses a reduced standard basis as its starting point und the strict transform may be computed on the level of the strict transforms of the elements of a reduced standard basis, we immediately see that the d_i and the e_i in the construction are unchanged and the hypersurfaces of maximal contact may be chosen to be the strict transforms of the previous ones. By direct calculation of the weak transform of J_k, we can now check that the two ideals indeed coincide.

q.e.d.

Using the previous lemmata, we can now state a modified resolution invariant, which implicitly uses the Hilbert-Samuel function as the first building block, but only requires the explicit computation of its maximal value upon each drop of maximal order of an auxilliary ideal:

$$(ord(J_k), n_1; ord(Coeff^{new}), n_2; ord(Coeff^V), n_3; \dots)$$

in contrast to a Villamayor-style invariant

$$(ord(\mathcal{I}_{X,w}), n_1; ord(Coeff^V), n_2; \dots)$$

or a Bierstone-Milman-style invariant

$$(HS(\mathcal{I}_{X,w}), n_1; ord(Coeff^{BM}), n_2; \dots),$$

where in each case the n_i denote counting of certain exceptional divisors and *Coeff* should be seen as a symbolic notation for a coefficient ideal of Villamayor and Bierstone-Milman respectively.
From a theoretical point of view, the only advantage of this new resolution invariant lies in the fact that we may now treat the use of the strict transform in the framework of order reduction. From the practical point of view, on the other hand, this modified invariant allows us to exploit order reduction of higher order generators of the ideal speeding up the resolution process in a large class of examples without trading the fewer blowing ups for the huge computations necessary to determine the Hilbert-Samuel stratum in each step.

5 Computational Aspects

For explicit calculations, it is most convenient to pass to an affine open covering of our smooth equidimensional scheme W and consider $\mathcal{I}_X(U) \subset \mathcal{O}_W(U)$ on each of the affine open sets. For simplicity of presentation, we will assume from now on that $\mathcal{O}_W(U) = K[y_1, \ldots, y_n]$.[4] To define the modified coefficient ideal, which allows the use of the strict instead of the weak transform, our construction will proceed by iteration of two steps: We first need to determine (the next variety in) a suitable flag in W and (the next parameter in) a regular system of paramters $x_1, \ldots, x_n$ for W subordinate to this flag. With respect to a local degree ordering on the set of monomials $Mon(x_1, \ldots, x_n)$, we determine (an intermediate result up to the current degree of) a reduced standard basis of $f_1, \ldots, f_k \in \mathcal{I}_X(U) := I \subset K[x_1, \ldots, x_n]$. This standard basis provides the complete information about the Hilbert-Samuel function through combinatorial reasoning on the staircase; the coefficients of the f_i w.r.t. to an appropriate subset of $\{x_1, \ldots, x_i\} \subset \{x_1, \ldots, x_n\}$ will subsequently be used to compute the modified coefficient ideal.

More precisely, the construction of the flag and the standard basis can be stated as follows:

Step0: Initialization

Let d_1 be the order of the ideal I. Create a copy I_{orig} of I for later use.

Step1: Find appropriate hypersurfaces

As the order of I is d_1, the order of $\Delta^{d_1-1}(I)$ is one and any order-1-element thereof (subject to the appropriate normal crossing conditions) provides a hypersurface of maximal contact for I. If necessary cover the

[4] The same construction can also be carried out in the case of $\mathcal{O}_W(U) = K[y_1, \ldots, y_n]/J$ for some ideal J. In that case, however, it may be necessary to pass to yet another open cover of U such that the local system of parameters for W at each point of the fixed smaller open set can be induced by the same set of elements. Even given such a set inducing the local systems of parameters on the whole open set, the subsequent computations still become by far more technical, obstructing the view to the heart of the considerations.

affine chart by finitely many open sets V_i such that on each of these we may use the same hypersurface at all points of the locus of maximal order $V(\Delta^{d_1-1}(I))$. This hypersurface gives rise to our first coordinate y_1. Similarly[5], we can proceed to determine $y_2,\ldots,y_{e_1}$, if there are further generators of order 1 of $\Delta^{d_1-1}(I)$.
As we will use these new y_i as the first elements of the set inducing at each point of the Hilbert-Samuel stratum a local regular system of parameters, it is convenient to use them as new variables replacing existent ones, be it directly, by passing to an open covering, by finite extension of our ground field or an appropriate combination of these methods.

Step2: do SB in 'degree' d_1

Now we express (at least) the order d_1 terms of the generators of I in terms of these new y_i. We then proceed through the standard basis algorithm in this degree d_1 in the sense that we form spolys of all pairs arising from generators in this degree and reduce them w.r.t. these generators until they are either reduced or themselves of order at least d_1+1.

Step3: proceed to next degree
We note for all generators of I_{orig} which appeared as degree d_1 generators of I that the respective degree is d_1. Then we replace all order d_1 generators of I by all possible products of one such generator with one of the already chosen y_j. Hence the new ideal I arising in this way has order d_1+1, with which we can return to steps 1 and 2 with the following modifications: the previously chosen y_j stay unchanged and the next d_k is defined, if additional y_j arise – in addition to newly formed spolys also older ones are reduced, whereever necessary. Step 3 can then be applied using the new d_j and we return to step 1 again unless we have just marked the last generator of I_{orig}.
If all generators of I_{orig} are marked, then we have found all contributions to the modified coefficient ideal, because all further standard basis element only provide coefficients which are combinations of coefficients already provided by lower order generators of I_{orig}. This is sufficient for our purposes and we can hence stop at this point. (Note that this is not the complete standard basis computation and hence we cannot detect the whole Hilbert-Samuel function from it, but only the first entries up to degree d_k.)

Step4: form modified coefficient ideal (cf. section 3)
On each chart which arose in the construction, we have now determined e_k new smooth hypersurfaces $V(y_1),\ldots,V(y_{e_k})$ and add $n-e_k$ further ones such that it gives rise to a regular system of parameters at each point of $V(\langle y_1,\ldots,y_{e_k}\rangle)$. With respect to these y_j we can then determine the modified coefficient ideal.
Note that for this coefficient ideal, we can then proceed as in the original algorithm of Villamayor.

[5]To this end, reduce $\Delta^{d_1-1}(I)$ w.r.t. the new y_1 and subsequently consider its order again.

6 Examples

The first example is mostly intended to illustrate what the respective ideals look like and how they are transformed.
To see that the modified approach can really contribute to an improvement of the performance of the resolution algorithm for certain classes of ideals, we subsequently state a very simple explicit example for which we have an immediate improvement of the maximal value of the Hilbert-Samuel function of the strict transform, whereas the maximal order of the weak transform stays 2.

6.1 $V(z^2 + x^3y^3, w^5 + x^5 + v^3y^2) \subset \mathbb{A}^5_{\mathbb{C}}$

In this case the ideal to be considered is

$$I = \langle z^2 + x^3y^3, w^5 + x^5 + v^3y^2\rangle$$

which is already a standard basis w.r.t. a local degree reverse lexicographical ordering on $Mon(x, y, z, w, v)$.
The first auxilliary ideal $\Delta(I) = \langle z, x^2y^3, x^3y^2, w^4, x^4, v^3y, v^2y^2\rangle$ is obviously of order 1 and $dim_{\mathbb{C}}(\Delta(I)/\langle x, y, z, w, v\rangle^2) = 1$. We thus choose the first hypersurface of maximal contact to be defined by $y_1 := z$. As this is already one of our coordinates, we do not need any coordinate change at this point.
Following through the algorithmic steps of section 4, the next interesting degree is 5, where our corresponding ideal J has the structure:

$$J = \langle z^5 + z^3x^3y^3, w^5 + x^5 + v^3y^2\rangle,$$

which is of order 5 and allows the hypersurfaces of maximal contact $V(x)$, $V(y)$, $V(z)$, $V(w)$, $V(y)$ already implying that the upcoming center should be the origin.
We now consider the situation after the blowing up at the origin in the various charts which we label (for convenience of the reader) by the generator of the exceptional divisor on this chart:

Chart 1: $E = V(x)$

$I_{strict} = \langle z^2 + x^4y^3, w^5 + 1 + v^3y^2\rangle$ which can easily be checked to allow at most order 1 at all points of this chart. In particular, the order and hence the Hilbert-Samuel function have decreased under this blowing up. The upcoming center will be determined inside the hypersurface $V(w^5 + 1 + v^3y^2)$. (The weak transform of J is $\langle z^5 + z^3x^4y^3, w^5 + 1 + v^3y^2\rangle$ which shows that its order has also dropped as was to be expected.)
For comparison, we now observe that $I_{weak} = \langle z^2 + x^4y^3, x^3w^5 + x^3 + x^3v^3y^2\rangle$ where the maximal order is still 2 and the order of the second generator has not decreased as much as in the case of the strict transform. Leaving us with the first hypersurface of maximal contact being $V(z)$ and a coefficient ideal of order 3 arising in this step.

Chart 2: $E = V(y)$

$I_{strict} = \langle z^2 + x^3y^4, w^5 + x^5 + v^3\rangle$ which is still of order 2 at the

zero locus of the ideal $\Delta(I) = \langle z, x^2y^4, x^3y^3, w^4, x^4, v^2\rangle$ that is along the line $V(x,z,w,v)$. But the drop of order of the second generator causes a decrease of the maximal value of the Hilbert-Samuel, which is now $(1,5,14,29,\dots)$ along the line $V(x,z,v,w)$ as compared to $(1,5,14,30,\dots)$ at the previous origin. The weak transform of J is $\langle z^5 + z^3x^3y^4, w^5 + x^5 + v^3\rangle$ which shows that its order has also dropped as was to be expected.
For comparison, we observe here that $I_{weak} = \langle z^2 + x^3y^4, y^3w^5 + y^3x^5 + v^3y^4\rangle$ where the maximal order is still 2, but the order of the second generator has even increased producing a coefficient ideal of order 7 (before splitting into monomial part y^3 and a non-monomial part of order 4).

Chart 3: $E = V(z)$
$I_{strict} = \langle 1 + x^3y^3z^4, w^5 + x^5 + v^3y^2\rangle$ of order at most 1 (analogous to the first chart). $I_{weak} = \langle 1 + x^3y^3z^4, w^5z^2 + x^5z^2 + v^3y^2z^2\rangle$ also of order 1.

Chart 4: $E = V(w)$
$I_{strict} = \langle z^2 + x^3y^3w^4, 1 + x^5 + v^3y^2\rangle$ of order at most 1 (again analogous to the first chart). $I_{weak} = \langle z^2 + x^3y^3w^4, w^3 + x^5w^3 + v^3y^2w^3\rangle$ of order 2 giving rise to a coefficient ideal of order 3.

Chart 5 $E = V(v)$
$I_{strict} = \langle z^2 + x^3y^3v^4, w^5 + x^5 + y^2\rangle$ of order 2, with a drop in the maximal value of the Hilbert-Samuel function to $(1,5,13,25,\dots)$ (analogous to the second chart). $I_{weak} = \langle z^2 + x^3y^3v^4, w^5v^3 + x^5v^3 + y^2v^3\rangle$ of order 2 giving rise to a coefficient ideal of order 5 (before splitting into a monomial part v^3 and a non-monomial part of order 2).

In this example, the order of the ideal was only two, to keep all computations at a level of complexity which can still be followed without difficulty. This also made sure that forming the usual coefficient ideals of the weak transforms the highest powers of ideals which needed to be computed were second powers.
If, however, the order of the original ideal is higher, the appearance of a factorial of the previous order in the exponents easily leads to far higher powers in the construction of the coefficient ideal which is of course iterated several times in Villamayor's approach. In our proposed approach we make sure that we descend the maximal possible number of hypersurfaces of maximal order at the very first descend of ambient dimension, thus minimizing the effect of taking iterated factorials.

6.2 $V(x^5 + y^{11}, z^9 + x^9) \subset \mathbb{A}^3$

As monomial ordering we choose a negative degree reverse lexicographical ordering with $z > y > x$ implying that the given generators already form a standard basis. Now the obvious choice of center is $V(x,y,z)$. Considering the strict and weak transforms as before, we now focus on the charts in which the exceptional divisors are $E = V(y)$ and $E = V(z)$ respectively, omitting the discussion of the third chart:

Chart 1: $E = V(z)$

$I_{strict} = \langle x^5 + y^{11}z^6, 1 + x^9 \rangle$ is already non-singular, $I_{weak} = \langle x^5 + y^{11}z^6, z^4 + x^9 z^4 \rangle$ leads to a coefficient ideal of order 8 (before splitting into monomial and non-monomial part) requiring up to 8-th powers of certain ideals in the computation.

Chart 2: $E = V(y)$

$I_{strict} = \langle x^5 + y^6, z^9 + x^9 \rangle$ leads by our algorithm to a locus of maximal Hilbert-Samuel function $V(x, y, z)$ and will be resolved after the subsequent blowing up; $I_{weak} = \langle x^5 + y^6, y^4 z^9 + y^4 x^9 \rangle$ gives rise to a coefficient ideal of order 10 requiring up to 60-th powers of certain ideals in the computation.

Remark 7 *As can be seen from the two previous examples, improvements of the choice of centers arise from the new variant of the algorithm, whenever the lowest order generator of the given ideal is harder to resolve than another one, which is of significantly higher order, but 'simpler' structure. Thus a heuristic choosing between the two approaches could take into account:*

- *number of generators of the ideal*
- *degrees of generators of the ideal*
- *number of terms in lowest order generators.*

In the presence of two CPUs, another way of joining the two approaches could be to start them parallely each on one CPU and interrupt the slower one as soon as the faster one returned a result. These lines of thought, however, are beyond the scope of this short article and have not been explored systematically up to now.

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